Fault Detection and Isolation for Linear Structured Systems

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Overview

Geometric Approach to The FDI Problem for Linear Systems

- $\Sigma : \begin{cases} \dot{x} &= Ax + Lf \\ y &= Cx \end{cases}$, $x \in \mathbb{R}^n$, $f \in \mathbb{R}^q$, $y \in \mathbb{R}^p$. Denoted by (A, L, C).

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  such that $\{C\mathcal{V}_i\}_{i=1}^{q}$ are **independent**, i.e., $C\mathcal{V}_i \neq \{0\}$ and $C\mathcal{V}_i \cap C\mathcal{V}_j = \{0\}\ \forall i \neq j$, where $\mathcal{V}_i$ is the smallest $(A + GC)-$invariant subspace containing im L_i.

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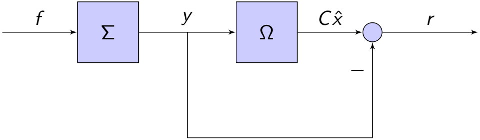

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- $e(t) \in \mathcal{V}_1 + \mathcal{V}_2 + \ldots + \mathcal{V}_q$ and $r(t) \in C\mathcal{V}_1 + C\mathcal{V}_2 + \ldots + C\mathcal{V}_q$.

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- Indeed, $r_i \neq 0$ **only if** $f_i \neq 0$, i.e., $r_i \neq 0$ implies that the ith fault occurs.
- The FDI problem is **solvable** for (A, L, C) if $\exists G$ such that $\{C\mathcal{V}_i\}_{i=1}^q$ are independent.

The FDI Problem for Linear Structured Systems

- The FDI problem is solvable for (A, L, C) if $\{C\mathcal{S}_i^*\}_{i=1}^{q}$ **independent**, where $\mathcal{S}_i^*$ is the smallest $(C, A)-$invariant subspace containing im L_i.[1]

[1] M. -A. Massoumnia (1986), 'A Geometric Approach to The Synthesis of Failure Detection Filters.'

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- In many scenarios, the exact values of entries in A, L and C are not known, but some **patterns** of A, L and C are known exactly.

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For example:

$$A = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 \\ c_2 & \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & c_3 & c_4 & \lambda_3 & 0 \\ c_5 & 0 & 0 & \lambda_4 & c_6 \\ 0 & 0 & c_7 & 0 & c_8 \end{bmatrix}, \quad L = \begin{bmatrix} c_9 & 0 \\ \lambda_5 & c_{10} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & c_{11} & 0 \\ 0 & 0 & 0 & \lambda_6 & \lambda_7 \\ 0 & 0 & 0 & c_{12} & c_{13} \end{bmatrix},$$

where $c_1, c_2, \ldots, c_{13}$ are nonzero real numbers, and $\lambda_1, \lambda_2, \ldots, \lambda_7$ are arbitrary real numbers.

Pattern Matrices and Pattern Classes

- Given a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p \times q}$, we define the *pattern class* of $\mathcal{M}$ as [2]

$$\mathcal{P}(\mathcal{M}) := \{M \in \mathbb{R}^{p \times q} \mid M_{ij} = 0 \text{ if } \mathcal{M}_{ij} = 0, \quad M_{ij} \neq 0 \text{ if } \mathcal{M}_{ij} = *.\}$$

[2] J. Jia, H. J. van Waarde, H. L. Trentelman, M. K. Camlibel (2020), 'A Unifying Framework for Strong Structural Controllability'.

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The FDI Problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$

Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$, $\mathcal{L} \in \{0, *, ?\}^{n \times q}$ and $\mathcal{C} \in \{0, *, ?\}^{p \times n}$. We call the family of systems of the form

$$\begin{aligned} \dot{x} &= Ax + Lf \\ y &= Cx, \end{aligned} \tag{1}$$

where $A \in \mathcal{P}(\mathcal{A})$, $L \in \mathcal{P}(\mathcal{L})$ and $C \in \mathcal{P}(\mathcal{C})$, **the linear structured system associated with $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{C}$**, represented by $(\mathcal{A}, \mathcal{L}, \mathcal{C})$.

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  1. The FDI problem is solvable for **at least one** member of a given structured system.[3]

[3] C. Commault, J.M. Dion, O. Sename and R. Motyeian (2000), 'Fault Detection and Isolation of Structured Systems.'

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- Research directions of the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$:
  1. The FDI problem is solvable for **at least one** member of a given structured system.[3]
  2. The FDI problem is solvable for **all** members of a given structured system.[4]

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[4] P. Rapisarda, A. R. F. Everts and M. K. Camlibel (2015), 'Fault Detection and Isolation for Systems Defined over Graphs.'

Problem Formulation

- Define the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ to be **solvable** if the FDI problem is solvable for every $(A, L, C) \in (\mathcal{A}, \mathcal{L}, \mathcal{C})$.

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- For **general** $\mathcal{A} \in \{0, *, ?\}^{n\times n}$, $\mathcal{L} \in \{0, *, ?\}^{n\times q}$ and $\mathcal{C} \in \{0, *, ?\}^{p\times n}$, the conditions for solvability of the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ are still **absent**.

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- **Problem:** Given $(\mathcal{A}, \mathcal{L}, \mathcal{C})$, find conditions under which the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable.

Conditions for Solvability of The FDI Problem for (A, L, C)

A Necessary and Sufficient Condition for Solvability of The FDI Problem for (A, L, C)

Let d_i be a positive integer such that

$$CA^j L_i = 0 \text{ for } j = 0, 1, \ldots, d_i - 2 \quad \text{and} \quad CA^{d_i - 1} L_i \neq 0.$$

If this d_i exists, we then call it the **index** of (A, L_i, C).

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Theorem 1

Consider the system (A, L, C) of the form (1). The FDI problem for (A, L, C) is solvable if and only if the index d_i exists for $i = 1, 2, \ldots, q$, and the matrix R has full column rank, where R is defined by

$$R := \begin{bmatrix} CA^{d_1-1}L_1 & CA^{d_2-1}L_2 & \cdots & CA^{d_q-1}L_q \end{bmatrix}. \quad (2)$$

Algebraic Conditions for Solvability of The FDI Problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$

Operations of Pattern Matrices

Table: Addition and multiplication within the set $\{0, *, ?\}$. [5]

$+$	0	$*$	?
0	0	$*$	?
$*$	$*$	?	?
?	?	?	?

$\cdot$	0	$*$	?
0	0	0	0
$*$	0	$*$	?
?	0	?	?

[5] B. Shali (2019), 'Strong Structural Properties of Structured Linear Systems.'

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Let $\mathcal{M} \in \{0, *, ?\}^{r \times s}$ and $\mathcal{N} \in \{0, *, ?\}^{s \times t}$. Define $\mathcal{M}\mathcal{N} \in \{0, *, ?\}^{r \times t}$ by

$$(\mathcal{M}\mathcal{N})_{ij} := \sum_{k=1}^{q} (\mathcal{M}_{ik} \cdot \mathcal{N}_{kj}) \quad i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, t.$$

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If $r = s$, we call $\mathcal{M}$ a square pattern matrix. Define the kth power $\mathcal{M}^k$ recursively by $\mathcal{M}^0 = \mathcal{I}, \quad \mathcal{M}^i = \mathcal{M}^{i-1}\mathcal{M}, \quad i = 1, 2, \ldots, k.$

Algebraic Conditions for Solvability of The FDI Problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$

Let η_i be a positive integer such that $\mathcal{C}\mathcal{A}^j\mathcal{L}_i = \mathcal{O}$ for $j = 0, 1, \ldots, \eta_i - 2$ and $\mathcal{C}\mathcal{A}^{\eta_i - 1}\mathcal{L}_i \neq \mathcal{O}$. If η_i exists, then we call it the **index** of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$.

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The relationship between the index of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ and that of $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$.

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The relationship between the index of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ and that of $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$.

Lemma 2

Consider the pattern matrix triple $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. Then the following holds:

1. *Let $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. If* **both** *the index η_i of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ and the index d_i of (A, L_i, C)* **exist**, *then $\mathbf{d_i} \geqslant \eta_\mathbf{i}$.*
2. *Suppose that the index η_i of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ exists, and suppose further that* **at least one** *entry of $\mathcal{C}\mathcal{A}^{\eta_i - 1}\mathcal{L}_i$ is equal to $*$. Let $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. Then, the index d_i of (A, L_i, C) exists and $\mathbf{d_i} = \eta_\mathbf{i}$.*
3. *If the index of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ does not exist, then the index of (A, L_i, C) does not exist for any $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$.*

An Example for Lemma 2

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathcal{L} = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} ? & * & 0 \\ 0 & * & 0 \end{bmatrix}. \tag{3}$$

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$$\mathcal{C}\mathcal{L}_1 = \begin{bmatrix} ? \\ 0 \end{bmatrix} \neq \mathcal{O}, \mathcal{C}\mathcal{L}_2 = \begin{bmatrix} * \\ * \end{bmatrix} \neq \mathcal{O},$$

$$\mathcal{C}\mathcal{A}^k\mathcal{L}_3 = \mathcal{O} \text{ for } k = 0, 1, 2, \dots.$$

This implies that $\eta_1 = \eta_2 = 1$ and the index of $(\mathcal{A}, \mathcal{L}_3, \mathcal{C})$ not exists.

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$$A = \begin{bmatrix} 0 & 0 & 0 \\ c1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ L = \begin{bmatrix} c_2 & 0 & 0 \\ 0 & c_3 & 0 \\ 0 & c_4 & c_5 \end{bmatrix}, \ C = \begin{bmatrix} \lambda_1 & c_6 & 0 \\ 0 & c_7 & 0 \end{bmatrix},$$

An Example for Lemma 2

$$A = \begin{bmatrix} 0 & 0 & 0 \\ c1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ L = \begin{bmatrix} c_2 & 0 & 0 \\ 0 & c_3 & 0 \\ 0 & c_4 & c_5 \end{bmatrix},\ C = \begin{bmatrix} \lambda_1 & c_6 & 0 \\ 0 & c_7 & 0 \end{bmatrix},$$

$$\begin{bmatrix} CL_1 & CL_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 c_2 & c_3 c_6 \\ 0 & c_3 c_9 \end{bmatrix}, \quad CAL_1 = \begin{bmatrix} c_1 c_2 c_6 \\ c_1 c_2 c_7 \end{bmatrix}.$$

$$CA^k L_3 = 0 \text{ for } k = 0, 1, \ldots$$

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- $d_1 \geqslant \eta_1$: If $\lambda_1 = 0$ then $d_1 = 2 > \eta_1$ and otherwise $d_1 = 1 = \eta_1$;
- $d_2 = 1 = \eta_2$;
- The index of (A, L_3, C) does not exist.

Algebraic Conditions for Solvability of The FDI Problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$

A necessary condition for solvability of the FDI problem:

Theorem 3

Consider the system $(\mathcal{A}, \mathcal{L}, \mathcal{C})$. Suppose that the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable. Then, the index η_i of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ exists for all $i = 1, 2, \ldots q$.

Algebraic Conditions for Solvability of The FDI Problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$

- In the sequel, we will assume that for all $i = 1, 2, \ldots q$ the indices η_i exist.

Algebraic Conditions for Solvability of The FDI Problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$

- In the sequel, we will assume that for all $i = 1, 2, \ldots q$ the indices η_i exist.
- Define the following pattern matrix associated with $(\mathcal{A}, \mathcal{L}, \mathcal{C})$:

$$\mathcal{R} := \begin{bmatrix} \mathcal{C}\mathcal{A}^{\eta_1 - 1}\mathcal{L}_1 & \mathcal{C}\mathcal{A}^{\eta_2 - 1}\mathcal{L}_2 & \cdots & \mathcal{C}\mathcal{A}^{\eta_q - 1}\mathcal{L}_q \end{bmatrix}. \quad (4)$$

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- We say that $\mathcal{R}$ has **full column rank** if all the matrices in the pattern class $\mathcal{P}(\mathcal{R})$ have full column rank.

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- We say that $\mathcal{R}$ has **full column rank** if all the matrices in the pattern class $\mathcal{P}(\mathcal{R})$ have full column rank.

Theorem 4

Consider the system $(\mathcal{A}, \mathcal{L}, \mathcal{C})$. Let $\mathcal{R}$ be the pattern matrix given by (4). The FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable if $\mathcal{R}$ has full column rank.

An Example for Theorem 4

$$\mathcal{A} = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & ? & 0 & ? & 0 \\ 0 & * & * & ? & 0 \\ * & 0 & 0 & ? & * \\ 0 & 0 & * & 0 & * \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} * & 0 \\ ? & * \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & * & * \end{bmatrix}.$$

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$$\begin{bmatrix} \mathcal{C}\mathcal{L}_1 & \mathcal{C}\mathcal{A}\mathcal{L}_1 \end{bmatrix} = \begin{bmatrix} 0 & * \\ 0 & ? \\ 0 & * \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{C}\mathcal{L}_2 & \mathcal{C}\mathcal{A}\mathcal{L}_2 & \mathcal{C}\mathcal{A}^2\mathcal{L}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & ? \\ 0 & 0 & * \end{bmatrix}$$

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$$\mathcal{A} = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & ? & 0 & ? & 0 \\ 0 & * & * & ? & 0 \\ * & 0 & 0 & ? & * \\ 0 & 0 & * & 0 & * \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} * & 0 \\ ? & * \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & * & * \end{bmatrix}.$$

$$\begin{aligned} \mathcal{R} &= \begin{bmatrix} \mathcal{C}\mathcal{A}\mathcal{L}_1 & \mathcal{C}\mathcal{A}^2\mathcal{L}_2 \end{bmatrix} \\ &= \begin{bmatrix} * & 0 \\ ? & ? \\ * & * \end{bmatrix}. \end{aligned}$$

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$$\mathcal{A} = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & ? & 0 & ? & 0 \\ 0 & * & * & ? & 0 \\ * & 0 & 0 & ? & * \\ 0 & 0 & * & 0 & * \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} * & 0 \\ ? & * \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & * & * \end{bmatrix}.$$

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Therefore, the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable.

Graph-theoretic Conditions for Solvability of The FDI Problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$

Associated Graphs of Pattern Matrices

- Given a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ $(p \leqslant q)$, we define the associated graph $G(\mathcal{M}) = (V, E)$ as follows:
- Node set $V = \{1, 2, \ldots, q\}$.
- Edge set $E = E_* \cup E_?$, where $E_* = \{(i, j) \in V \times V \mid \mathcal{M}_{ji} = *\}$ and $E_? = \{(i, j) \in V \times V \mid \mathcal{M}_{ji} = ?\}$.

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$$\mathcal{M} = \begin{bmatrix} * & ? & * \\ 0 & ? & * \end{bmatrix}$$

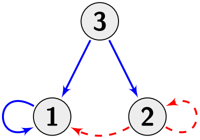

Figure: The graph $G(\mathcal{M})$.

Colorability of Graphs Associated with Pattern Matrices

- Consider a graph $G(\mathcal{M})$ with $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ $(p \leqslant q)$.
  1. Initially, color all nodes of $G(\mathcal{M})$ white.
  2. If a node $i \in V$ (of any color) has
     - **exactly one white out-neighbor** j and
     - $(i, j) \in E_*$,

     we change the color of j to black.
  3. Repeat the Step 2 until no more color changes are possible.
- The $G(\mathcal{M})$ is called **colorable** if all the nodes in $\{1, 2, \ldots, p\}$ are colored black finally.

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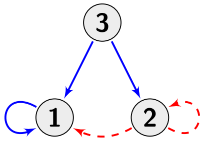

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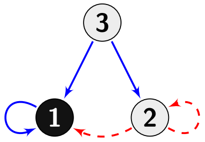

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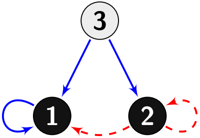

A Graph-theoretic Condition for Solvability of The FDI Problem

Define the **transpose** of $\mathcal{R}$ as the pattern matrix $\mathcal{R}^\top \in \{0, *, ?\}^{s \times r}$ with $(\mathcal{R}^\top)_{ij} = \mathcal{R}_{ji}$ for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, r$.

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Theorem 5

Consider the system $(\mathcal{A}, \mathcal{L}, \mathcal{C})$. Suppose that the indices η_i exists for $i = 1, 2, \ldots, q$. Let $\mathcal{R}$ be the pattern matrix given by (4). Then, the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable if $G(\mathcal{R}^\top)$ is colorable.

An Example for Theorem 5

$$\mathcal{A} = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & ? & 0 & ? & 0 \\ 0 & * & * & ? & 0 \\ * & 0 & 0 & ? & * \\ 0 & 0 & * & 0 & * \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} * & 0 \\ ? & * \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & * & * \end{bmatrix}.$$

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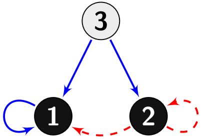

Figure: The graph $G(\mathcal{R}^\top)$ is colorable.

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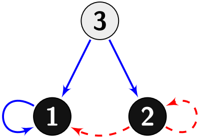

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Summary

- Formalize the solvability of the FDI problem for linear structured systems.

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Summary

- Formalize the solvability of the FDI problem for linear structured systems.
- Establish a necessary and sufficient condition for solvability of the FDI problem for a given particular LTI system.
- Establish a sufficient condition for solvability of the FDI problem in terms of a rank test on a pattern matrix associated with the structured system.
- Establish a graph-theoretic condition for solvability of the FDI problem using the concept of colorability of a graph.

For Further Reading I

J. Jia, H. L. Trentelman and M. K. Camlibel (2020).
Fault Detection and Isolation for Linear Structured Systems.
https://arxiv.org/abs/2003.01502

Thank You for Your Attention!
The End