A Unifying Framework for Strong Structural Controllability

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Overview

1. Introduction
2. Problem formulation
3. Main results
4. Summary

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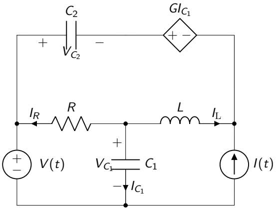

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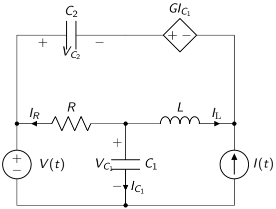

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$$A = \begin{bmatrix} -\frac{1}{C_1 R} & 0 & -\frac{1}{C_1} \\ 0 & 0 & -\frac{1}{C_2} \\ \frac{R-G}{RL} & \frac{1}{L} & -\frac{G}{L} \end{bmatrix}$$

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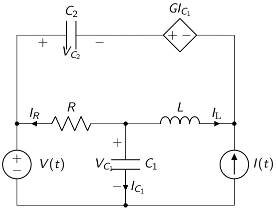

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Pattern matrices and Pattern Classes

- Given a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p \times q}$, we define the *pattern class* of $\mathcal{M}$ as

$$\mathcal{P}(\mathcal{M}) := \{M \in \mathbb{R}^{p \times q} \mid M_{ij} = 0 \text{ if } \mathcal{M}_{ij} = 0, \\ M_{ij} \neq 0 \text{ if } \mathcal{M}_{ij} = *.\}$$

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Strong Structural Controllability of $(\mathcal{A}, \mathcal{B})$

- Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices.
- If for every $A \in \mathcal{P}(\mathcal{A})$ and $B \in \mathcal{P}(\mathcal{B})$ the linear dynamical system

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is controllable, we call $(\mathcal{A}, \mathcal{B})$ is **strongly structurally controllable** (or **controllable** for short).

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- **Problem arises:** How can we check the controllability of $(\mathcal{A}, \mathcal{B})$?

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Problem formulation

- For **general** $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$, the conditions for controllability of $(\mathcal{A}, \mathcal{B})$ are still **absent**.

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- **Problem statement:**
  Let $\mathcal{A} \in \{0, *, ?\}^{n\times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n\times m}$ be pattern matrices.
  Can we provide conditions for controllability of $(\mathcal{A}, \mathcal{B})$ *both in* **algebraic** *and* **graph-theoretic** *terms ?*

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Definition 1

For a given pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p \times q}$, we say $\mathcal{M}$ **has full row rank** if the matrix M has full row rank for all $M \in \mathcal{P}(\mathcal{M})$.

Algebraic conditions for controllability of $(\mathcal{A}, \mathcal{B})$

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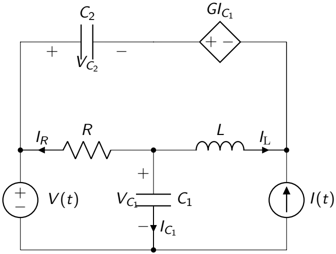
Theorem 1

Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices. Let $\bar{\mathcal{A}} \in \{0, *, ?\}^{n \times n}$ be the pattern matrix obtained from $\mathcal{A}$ by modifying the diagonal entries of $\mathcal{A}$ as follows:

$$\bar{\mathcal{A}}_{ii} := \begin{cases} * & \text{if } \mathcal{A}_{ii} = 0, \\ ? & \text{otherwise.} \end{cases} \quad (2)$$

The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both pattern matrices $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$ and $\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}$ have full row rank.

Example for algebraic conditions



$$\mathcal{A} = \begin{bmatrix} * & 0 & * \\ 0 & 0 & * \\ ? & * & * \end{bmatrix}, \; \mathcal{B} = \begin{bmatrix} * & 0 \\ 0 & * \\ ? & 0 \end{bmatrix}$$

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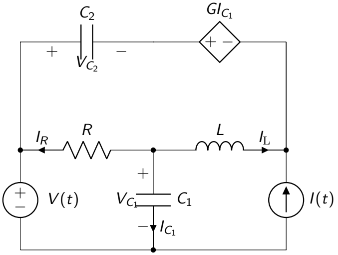

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$$[\mathcal{A} \quad \mathcal{B}] = \begin{bmatrix} * & 0 & * & \boxed{*} & 0 \\ 0 & 0 & * & 0 & \boxed{*} \\ ? & \boxed{*} & * & ? & 0 \end{bmatrix}$$

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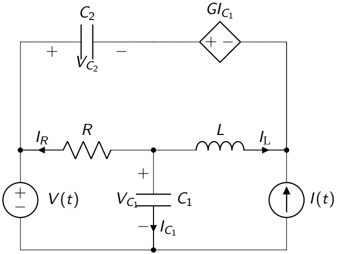

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Obviously, $(\mathcal{A}, \mathcal{B})$ is controllable.

Associated graphs of pattern matrices

- Given a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ $(p \leq q)$, we define the associated graph $G(\mathcal{M}) = (V, E)$ as follows:
- Node set $V = \{1, 2, \ldots, q\}$.
- Edge set $E = E_* \cup E_?$ where $E_* = \{(i, j) \in V \times V \mid \mathcal{M}_{ji} = *\}$ and $E_? = \{(i, j) \in V \times V \mid \mathcal{M}_{ji} = ?\}$.

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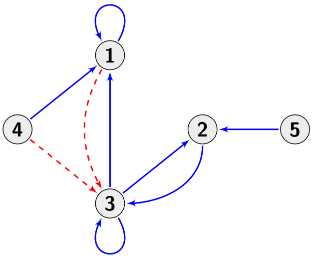

Figure: The graph $G(\mathcal{M})$.

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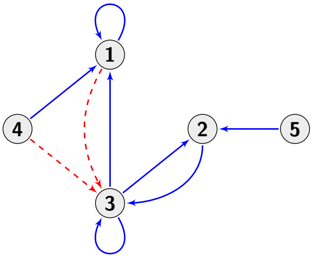

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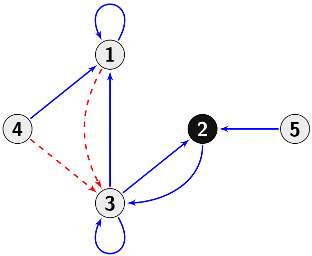

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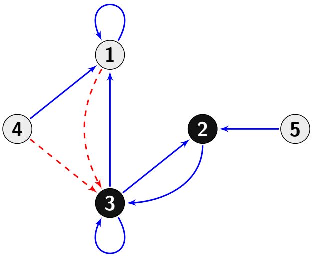

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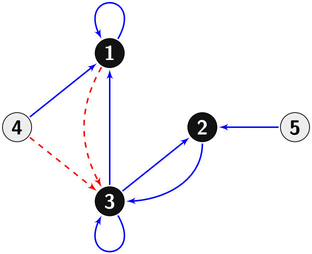

Theorem 2

Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ with $p \leq q$. The matrix $\mathcal{M}$ has full row rank if and only if the associated graph $G(\mathcal{M})$ is colorable.

Graph theoretic conditions for controllability of $(\mathcal{A}, \mathcal{B})$

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Theorem 1

Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices. Let $\bar{\mathcal{A}} \in \{0, *, ?\}^{n \times n}$ be defined as (2). The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both pattern matrices $\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix}$ and $\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}$ have full row rank.

Graph theoretic conditions for controllability of $(\mathcal{A}, \mathcal{B})$

Theorem 2

Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ with $p \leq q$. The matrix $\mathcal{M}$ has full row rank if and only if the associated graph $G(\mathcal{M})$ is colorable.

Theorem 1

Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices. Let $\bar{\mathcal{A}} \in \{0, *, ?\}^{n \times n}$ be defined as (2). The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both pattern matrices $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$ and $\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}$ have full row rank.

Theorem 3

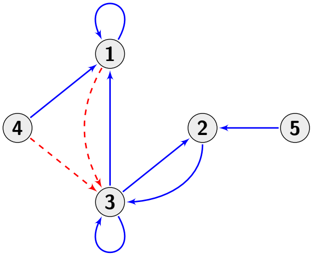
Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices. Let $\bar{\mathcal{A}} \in \{0, *, ?\}^{n \times n}$ be defined as (2). The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both $G(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix})$ and $G(\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix})$ are colorable.

Example for graph theoretic conditions

- Consider the electrical circuit:

$$[\mathcal{A} \quad \mathcal{B}] = \begin{bmatrix} * & 0 & * & * & 0 \\ 0 & 0 & * & 0 & * \\ ? & * & * & ? & 0 \end{bmatrix}$$

$$[\bar{\mathcal{A}} \quad \mathcal{B}] = \begin{bmatrix} ? & 0 & * & * & 0 \\ 0 & * & * & 0 & * \\ ? & * & ? & ? & 0 \end{bmatrix}$$



$G([\mathcal{A} \quad \mathcal{B}])$

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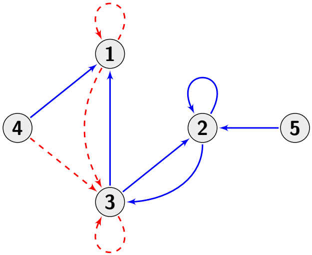

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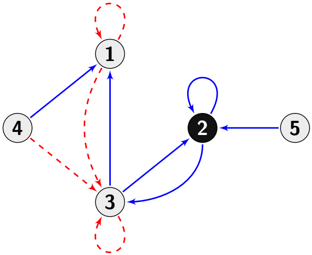

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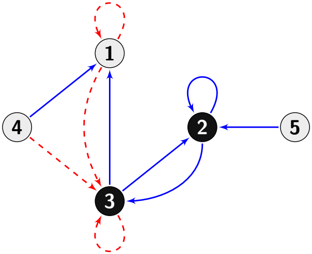

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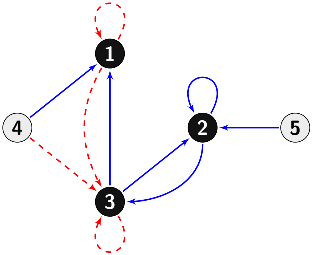

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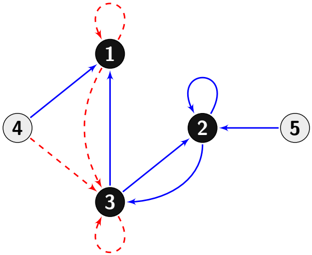

$G([\bar{\mathcal{A}} \quad \mathcal{B}])$

- Therefore, $(\mathcal{A}, \mathcal{B})$ is controllable.

1 Introduction

2 Problem formulation

3 Main results

4 Summary

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Summary

- $\{0, *\} \Rightarrow \{0, *, ?\}$.
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Summary

- $\{0, *\} \Rightarrow \{0, *, ?\}$.
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- $\mathcal{M}$ has full row rank $\Leftrightarrow$ $G(\mathcal{M})$ is colorable.
- $(\mathcal{A}, \mathcal{B})$ is controllable $\Leftrightarrow$ $G(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$ and $G(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ are colorable.

For Further Reading I

Jiajia Jia, Henk J. van Waarde, Harry L. Trentelman, M. Kanat Camlibel
A Unifying Framework for Strong Structural Controllability.
https://arxiv.org/abs/1903.03353.

Thank you for your attention!
The End