A Unifying Framework for Strong Structural Controllability

J. Jia, H. J. van Waarde, M. K. Camlibel, and H. L. Trentelman

Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence,
University of Groningen, The Netherlands

Benelux Meeting on Systems and Control, 2019

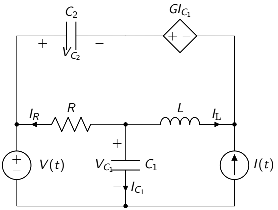
Overview

Controllability of Linear Systems

- For linear systems of the form

$$\dot{x} = Ax + Bu,$$

  controllability can be verified by Kalman rank test or the Hautus test.
- In many scenarios, the exact values of entries in A and B are not known, but some **patterns** of A and B are known exactly.



$x = [V_{C_1} \quad V_{C_2} \quad I_L]^T$

$u = [V(t) \quad I(t)]^T$

$$A = \begin{bmatrix} -\frac{1}{C_1 R} & 0 & -\frac{1}{C_1} \\ 0 & 0 & -\frac{1}{C_2} \\ \frac{R-G}{RL} & \frac{1}{L} & -\frac{G}{L} \end{bmatrix} \qquad \mathcal{A} = \begin{bmatrix} * & 0 & * \\ 0 & 0 & * \\ ? & * & * \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{RC_1} & 0 \\ 0 & -\frac{1}{C_2} \\ \frac{G-R}{RL} & 0 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} * & 0 \\ 0 & * \\ ? & 0 \end{bmatrix}$$

Pattern matrices and Pattern Classes

- Given a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p \times q}$, we define the *pattern class* of $\mathcal{M}$ as

$$\mathcal{P}(\mathcal{M}) := \{M \in \mathbb{R}^{p \times q} \mid M_{ij} = 0 \text{ if } \mathcal{M}_{ij} = 0, \\ M_{ij} \neq 0 \text{ if } \mathcal{M}_{ij} = *.\}$$

Remark: M_{ij} can be any real value, if $\mathcal{M}_{ij} = ?$.

$$\mathcal{M} = \begin{bmatrix} * & 0 & * \\ 0 & 0 & * \\ ? & * & * \end{bmatrix}$$

$$\mathcal{M} \in \{0, *, ?\}^{3 \times 3}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix} \in \mathcal{P}(\mathcal{M}) \qquad \begin{bmatrix} 1 & \boxed{1} & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix} \notin \mathcal{P}(\mathcal{M})$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \in \mathcal{P}(\mathcal{M}) \qquad \begin{bmatrix} \boxed{0} & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \notin \mathcal{P}(\mathcal{M})$$

Strong Structural Controllability of $(\mathcal{A}, \mathcal{B})$

- Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices.
- If for every $A \in \mathcal{P}(\mathcal{A})$ and $B \in \mathcal{P}(\mathcal{B})$ the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

  is controllable, we call $(\mathcal{A}, \mathcal{B})$ is **strongly structurally controllable** (or **controllable** for short).
- **Problem arises:** How can we check the controllability of $(\mathcal{A}, \mathcal{B})$?

Brief review of existing research

- $\mathcal{A} \in \{0, *\}^{n\times n}$ and $\mathcal{B} \in \{0, *\}^{n\times m}$:
  - Mayeda et. al. (1979), "*Strong Structural Controllability.*"
  - Reinschke et. al. (1992), "*On strong structural controllability of linear systems.*"
  - Jarczyk et. al. (2011), "*Strong structural controllability of linear systems revisited.*"
  - Chapman et. al. (2013), "*On strong structural controllability of networked systems: A constrained matching approach.*"
  - Trefois et. al. (2015), "*Zero forcing number, constrained matchings and strong structural controllability*."
- $\mathcal{A} \in \{0, *, ?\}^{n\times n}$ and $\mathcal{B} \in \{0, *\}^{n\times m}$:
  - Monshizadeh et. al. (2014), "*Zero Forcing Sets and Controllability of Dynamical Systems Defined on Graphs.*"

Problem formulation

- For **general** $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$, the conditions for controllability of $(\mathcal{A}, \mathcal{B})$ are still **absent**.

- **Problem statement:**
  Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices.
  Can we provide conditions for controllability of $(\mathcal{A}, \mathcal{B})$ both in **algebraic** *and* **graph-theoretic** *terms ?*

Algebraic conditions for controllability of $(\mathcal{A}, \mathcal{B})$

Definition 1

For a given pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p \times q}$, we say $\mathcal{M}$ **has full row rank** if the matrix M has full row rank for all $M \in \mathcal{P}(\mathcal{M})$.

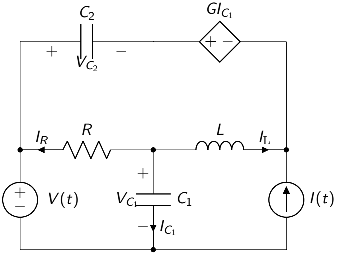
Theorem 1

Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices. Let $\bar{\mathcal{A}} \in \{0, *, ?\}^{n \times n}$ be the pattern matrix obtained from $\mathcal{A}$ by modifying the diagonal entries of $\mathcal{A}$ as follows:

$$\bar{\mathcal{A}}_{ii} := \begin{cases} * & \text{if } \mathcal{A}_{ii} = 0, \\ ? & \text{otherwise.} \end{cases} \tag{2}$$

The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both pattern matrices $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$ and $\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}$ have full row rank.

Example for algebraic conditions



$$\mathcal{A} = \begin{bmatrix} * & 0 & * \\ 0 & 0 & * \\ ? & * & * \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} * & 0 \\ 0 & * \\ ? & 0 \end{bmatrix}$$

$$[\mathcal{A} \quad \mathcal{B}] = \begin{bmatrix} * & 0 & * & \boxed{*} & 0 \\ 0 & 0 & * & 0 & \boxed{*} \\ ? & \boxed{*} & * & ? & 0 \end{bmatrix}$$

$$[\bar{\mathcal{A}} \quad \mathcal{B}] = \begin{bmatrix} ? & 0 & * & \boxed{*} & 0 \\ 0 & * & * & 0 & \boxed{*} \\ ? & \boxed{*} & ? & ? & 0 \end{bmatrix}$$

Obviously, $(\mathcal{A}, \mathcal{B})$ is controllable.

Associated graphs of pattern matrices

- Given a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ $(p \leq q)$, we define the associated graph $G(\mathcal{M}) = (V, E)$ as follows:
- Node set $V = \{1, 2, \ldots, q\}$.
- Edge set $E = E_* \cup E_?$ where $E_* = \{(i, j) \in V \times V \mid \mathcal{M}_{ji} = *\}$ and $E_? = \{(i, j) \in V \times V \mid \mathcal{M}_{ji} = ?\}$.

$$\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix} = \begin{bmatrix} * & 0 & * & * & 0 \\ 0 & 0 & * & 0 & * \\ ? & * & * & ? & 0 \end{bmatrix}$$

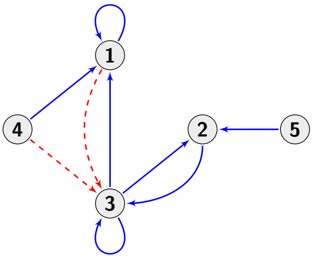


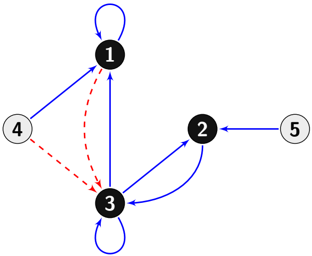
Figure: The graph $G(\mathcal{M})$.

Colorability of graphs associated with pattern matrices

- Consider a graph $G(\mathcal{M})$ with $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ $(p \leq q)$.
  1. Initially, color all nodes of $G(\mathcal{M})$ white.
  2. If a node $i \in V$ (of any color) has
     - **exactly one white out-neighbor** j and
     - $(i, j) \in E_*$,

     we change the color of j to black.
  3. Repeat the Step 2 until no more color changes are possible.
- The $G(\mathcal{M})$ is called **colorable** if all the nodes in $\{1, 2, \ldots, p\}$ are colored black finally.

$$\mathcal{M} = [\mathcal{A} \quad \mathcal{B}] = \begin{bmatrix} * & 0 & * & * & 0 \\ 0 & 0 & * & 0 & * \\ ? & * & * & ? & 0 \end{bmatrix}$$



Graph theoretic conditions for controllability of $(\mathcal{A}, \mathcal{B})$

Theorem 2

Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ with $p \leq q$. The matrix $\mathcal{M}$ has full row rank if and only if the associated graph $G(\mathcal{M})$ is colorable.

Theorem 1

Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices. Let $\bar{\mathcal{A}} \in \{0, *, ?\}^{n \times n}$ be defined as (2). The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both pattern matrices $\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix}$ and $\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}$ have full row rank.

Theorem 3

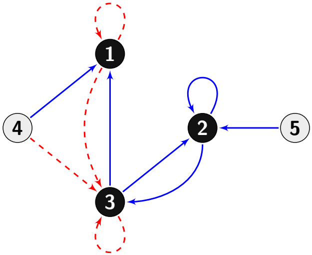
Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices. Let $\bar{\mathcal{A}} \in \{0, *, ?\}^{n \times n}$ be defined as (2). The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both $G(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$ and $G(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ are colorable.

Example for graph theoretic conditions

- Consider the electrical circuit:

$$[\mathcal{A} \quad \mathcal{B}] = \begin{bmatrix} * & 0 & * & * & 0 \\ 0 & 0 & * & 0 & * \\ ? & * & * & ? & 0 \end{bmatrix}$$

$$[\bar{\mathcal{A}} \quad \mathcal{B}] = \begin{bmatrix} ? & 0 & * & * & 0 \\ 0 & * & * & 0 & * \\ ? & * & ? & ? & 0 \end{bmatrix}$$



$G([\bar{\mathcal{A}} \quad \mathcal{B}])$

- Therefore, $(\mathcal{A}, \mathcal{B})$ is controllable.

Summary

- $\{0, *\} \Rightarrow \{0, *, ?\}$.
- $(\mathcal{A}, \mathcal{B})$ is controllable $\Leftrightarrow$ $\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix}$ and $\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}$ have full row rank.
- $\mathcal{M}$ has full row rank $\Leftrightarrow$ $G(\mathcal{M})$ is colorable.
- $(\mathcal{A}, \mathcal{B})$ is controllable $\Leftrightarrow$ $G(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$ and $G(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ are colorable.

For Further Reading I

Jiajia Jia, Henk J. van Waarde, Harry L. Trentelman, M. Kanat Camlibel
A Unifying Framework for Strong Structural Controllability.
https://arxiv.org/abs/1903.03353.

Thank you for your attention!
The End