Strong Structural Controllability of Systems on Colored Graphs

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Benelux Meeting on Systems and Control, 2018

Overview

Controllability of systems on graphs

- Consider a simple directed graph $\mathcal{G} = (V, E)$, $V = \{1, \ldots, n\}$ and $E \subset V \times V$.
- Qualitative class associated with $\mathcal{G}$:

$$\mathcal{Q}(\mathcal{G}) = \{A \in \mathbb{C}^{n \times n} \mid \text{for } i \neq j : A_{ji} \neq 0 \Leftrightarrow (i,j) \in E\}.$$

- The diagonal entries of $A \in \mathcal{Q}(\mathcal{G})$ do not depend on the structure of $\mathcal{G}$, and these are 'free elements' (0 or nonzero).
- Let $V_L = \{v_1, \ldots, v_m\} \subset V$ be the leader set. Associated input matrix $B = (e_{v_1}, \ldots, e_{v_m})$.
- The leader/follower system defined on graph $\mathcal{G}$: $\dot{x} = Ax + Bu$, with $A \in \mathcal{Q}(\mathcal{G})$.

Definition: $(\mathcal{G}, V_L)$ is called controllable if (A, B) is controllable for all $A \in \mathcal{Q}(\mathcal{G})$.

Zero forcing set and controllability of $(\mathcal{G}, V_L)$

Theorem (Monshizadeh, Zhang and Camlibel [1])
Let $\mathcal{G}$ be a simple directed graph and $V_L \subset V$. Then, $(\mathcal{G}, V_L)$ is controllable if and only if V_L is a zero forcing set.

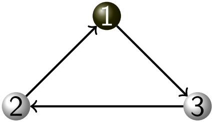

Figure: Simple directed graph $\mathcal{G}$.

- If v is a black vertex in $\mathcal{G}$ with exactly one white out-neighbor u, then we change the color of u to black, and write $v \rightarrow u$.

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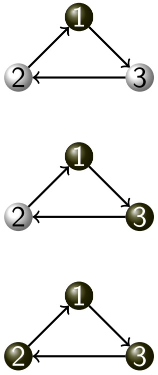


- If v is a black vertex in $\mathcal{G}$ with exactly one white out-neighbor u, then we change the color of u to black, and write $v \rightarrow u$.
- The set $\{1\}$ is a **zero forcing set** for the graph $\mathcal{G}$.

Example

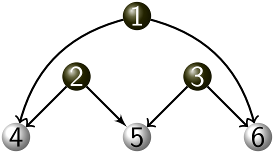


Figure: Graph $\mathcal{G}$ with $V_L = \{1, 2, 3\}$

- V_L is not a zero forcing set, which implies that $(\mathcal{G}, V_L)$ is not controllable.
- The situation changes drastically if we impose weights to be equal, but still nonzero.

Example

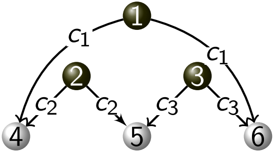
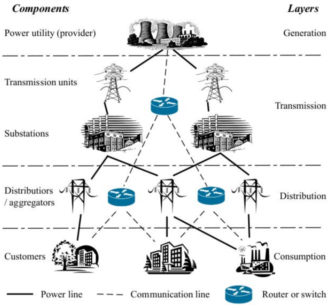


Figure: Graph $\mathcal{G}$ with $V_L = \{1, 2, 3\}$

- This graph turns out to be controllable, i.e., (A, B) is controllable for all A associated with this graph. This can be checked by the **PBH** test.
- Note that it becomes very complicated to use the **PBH** test if the graph is of large scale.

Motivation

- Physical: dependence of nonzero parameters
- Particular structure: symmetric, undirected or unweighted



(*Picture from "Secure and Efficient Capability-Based Power Management in the Smart Grid"*)

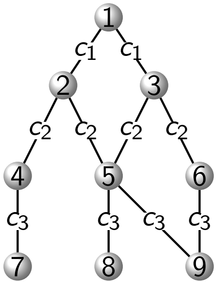


Figure: Underlying graph with identical edge weights.

Problem formulation

- **Colored simple directed graph** $\mathcal{G}(\pi) = (V, E, \pi)$ associated with a simple directed graph $\mathcal{G} = (V, E)$ and an edge partition $\pi = \{E_1, E_2, \ldots, E_k\}$, where $E_i \cap E_j = \emptyset$ for $i \neq j$, and edges have the same color if and only if they are in the same cell.
- **Colored qualitative class** of $\mathcal{G}(\pi) = (V, E, \pi)$: a family of matrices associated with $\mathcal{G}(\pi)$:

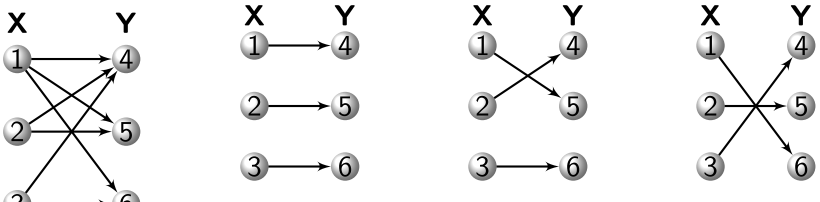
$$\mathcal{Q}_\pi(\mathcal{G}) = \{A \in \mathcal{Q}(\mathcal{G}) \mid A_{ji} = A_{lk} \text{ if } (i, j), (k, l) \in E_r \text{ for some } r\}.$$

- $(\mathcal{G}(\pi), V_L)$ **is called controllable if** (A, B) **is controllable for all** $A \in \mathcal{Q}_\pi(\mathcal{G})$**.**
- **Aim:** Establish graph theoretical tests for controllability of $(\mathcal{G}(\pi), V_L)$.

Overview

Bipartite graphs and Perfect matching

- A simple directed graph $\mathcal{G} = (V, E)$ is called **bipartite** if there exist two nonempty disjoint vertex sets X and Y such that $X \cup Y = V$ and $(i, j) \in E$ only if $i \in X$ and $j \in Y$, denoted by $G = (X, Y, E)$.
- A set of t edges $\mathcal{E} \subset E$ is called a t-**matching** in G, if no two edges in $\mathcal{E}$ share a vertex.
- In the special case of $|X| = |Y|$, such a t-matching is called a **perfect matching** if $t = |X|$.



p_1 p_2 p_3

Bipartite graphs and Perfect matching

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- In the special case of $|X| = |Y|$, such a t-matching is called a **perfect matching** if $t = |X|$.
- The pattern class $\mathcal{P}(G)$ of this bipartite graph G is defined as

$$\mathcal{P}(G) = \{M \in \mathbb{C}^{t \times s} | M_{ji} \neq 0 \Leftrightarrow (x_i, y_j) \in E\}.$$

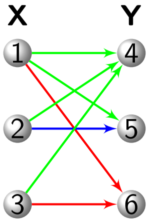
Colored bipartite graph and colored pattern class

- A colored bipartite graph $G(\pi) = (X, Y, E, \pi)$ is a bipartite graph with a partition of the edge set $\pi = \{E_1, E_1, \ldots, E_t\}$ with to each cell assigned a color.
- The pattern class of the colored bipartite graph $G(\pi)$ is defined by

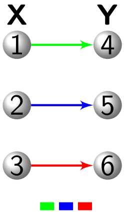
$$\mathcal{P}_\pi(G) = \{A \in \mathcal{P}(G) | A_{ji} = A_{lk} \\ \text{if } (x_i, y_j), (x_k, y_l) \in E_r \text{ for some } r\}.$$

- **Aim:** Suppose $|X| = |Y|$, find a condition to verify whether all $A \in \mathcal{P}_\pi(G)$ are nonsingular.
This condition is used to establish the graph-theoretic condition.

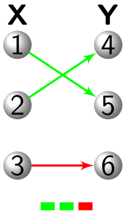
Equivalence classes of perfect matchings



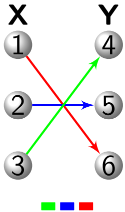
Colored bipartite graph.



$p_1, sgn(p_1) = 1$



$p_2, sgn(p_2) = -1$



$p_3, sgn(p_3) = -1$

Equivalence classes $\mathbb{P} = \{\mathbb{P}_1, \mathbb{P}_2\}$, where $\mathbb{P}_1 = \{p_1, p_3\}$ with $sgn(\mathbb{P}_1) = 0$, and $\mathbb{P}_2 = \{p_2\}$ with $sgn(\mathbb{P}_2) = -1$.

Lemma 1: Let $G(\pi) = (X, Y, E, \pi)$ be a colored bipartite graph and $|X| = |Y|$. Then, all $A \in \mathcal{P}_\pi(G)$ are nonsingular if and only if there exists exactly one equivalence class of perfect matchings having nonzero signature.

Colored color change rule and colored zero forcing set

Let $\mathcal{G}(\pi) = (V, E, \pi)$ be a colored simple graph, with each vertex colored either white or black. Let $C \subseteq V(G)$ be set of vertices initially colored black.

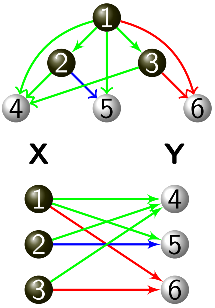
- Given $X \subset C$ and $Y = N_{V \setminus C}(X)$, assume $|X| = |Y|$. Note that the subgraph $G(X, Y, E_{XY}, \pi)$ is a colored bipartite graph. If **exactly one equivalence class of perfect matchings has nonzero signature** in $G(X, Y, E_{XY}, \pi)$. Then X forces Y to be black, and we write $X \overset{C}{\Rightarrow} Y$.
- The **derived set** $D_c(C)$ is the set of black vertices obtained by applying this color change rule until no more changes are possible. If $D_c(C) = V$, C is called a **colored zero forcing set**.

Example for colored zero forcing set

Theorem 2

Consider a colored simple directed graph $\mathcal{G}(\pi) = (V, E, \pi)$, with leader set $V_L \subset V$. Assume V_L is a colored zero forcing set. Then $(\mathcal{G}(\pi), V_L)$ is controllable.

$\mathcal{G}(\pi) = (V, E, \pi)$



$G(\pi) = (X, Y, E_{XY}, \pi)$

Note the colored bipartite graph $G(\pi)$ is same as the one in previous example.

Colored color change rule: $\{1, 2, 3\} \overset{C}{\Rightarrow} \{4, 5, 6\}$.

$V_L = \{1, 2, 3\}$ is a colored zero forcing set.

$(\mathcal{G}(\pi), V_L)$ is controllable by Theorem 2.

Main results

Lemma 1 Let $G(\pi) = (X, Y, E, \pi)$ be a colored bipartite graph and $|X| = |Y|$. Then, all $A \in \mathcal{P}_\pi(G)$ are nonsingular if and only if there exists exactly one equivalence class of perfect matchings having nonzero signature.

Theorem 2
Consider a colored simple directed graph $\mathcal{G}(\pi) = (V, E, \pi)$, with leader set $V_L \subset V$. Assume V_L is a colored zero forcing set. Then $(\mathcal{G}(\pi), V_L)$ is controllable.

Corollary 3
Consider a system defined on a colored simple directed graph $\mathcal{G}(\pi) = (V, E, \pi)$ with $|\pi| = |E|$ with leader set $V_L \subset V$. Then $(\mathcal{G}, V_L)$ is controllable if and only if V_L is a colored zero forcing set.

Note: In the special case $|\pi| = |E|$, our condition is equivalent to that in terms of zero forcing set.

Summary

- Extension to systems defined on graphs with identical edge weights, which is a other kind of dependencies between the nonzero parameters.
- Establish a graph theoretic test for controllability of system on colored graph in terms of colored zero forcing set.

For Further Reading I

N. Monshizadeh, S. Zhang, and M. K. Camlibel (2014).
Zero Forcing Sets and Controllability of Dynamical Systems Defined on Graphs.

Shima Sadat Mousaviy, Mohammad Haeriy, and Mehran Mesbahi (2017).
On the Structural and Strong Structural Controllability of Undirected Networks.

F. Liu, A. S. Morse (2017).
Structural Controllability of Linear Time-invariant Systems.

Thank you for your attention!
The End