A Sufficient Condition for Colored Strong Structural Controllability of Networks

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Introduction

Most of the literature on the controllability of networked systems is devoted to the controllability of leader/follower systems of the form

$$\dot{x} = Ax + Bu, \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $A \in \mathbb{R}^{n\times n}$ is the system matrix that represents the graph structure of the network, and $B \in \mathbb{R}^{n\times m}$ encodes the vertices (called leaders) through which the input signals enter the network. In one part of the literature, the analysis of network controllability has centered around structural controllability [1, 2].

In this paper, we consider leader/follower networked systems where some of the interconnections are constrained to have identical weights. Since graph theoretical results can reveal important structural properties hidden in matrix representations, in this paper our aim is to establish graph-theoretical conditions under which such networked systems are strongly structurally controllable.

Problem Formulation

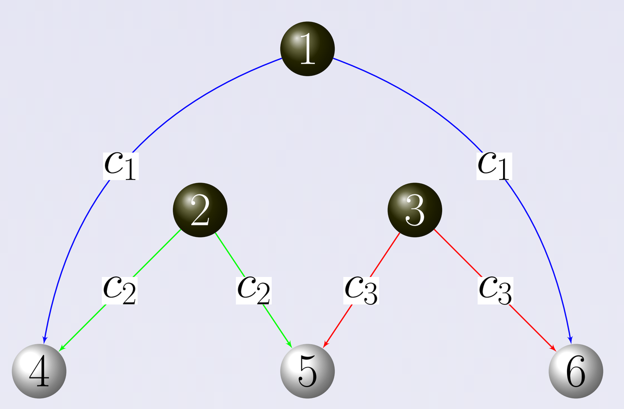
Let $\mathcal{G} = (V, E)$ be a simple directed graph. We will now formalize that certain off-diagonal entries in the matrices A in the qualitative class $\mathcal{Q}(\mathcal{G})$ are constrained to be equal. To do this, we introduce a partition

$$\pi = \{E_1, E_2, \ldots, E_k\}$$

of the edge set E. The edges in a given cell E_r are constrained to have identical weights. To visualize this, we then say that these edges have the same color: edges have the same color if and only if they are in the same cell E_r for some r. The colors will be denoted by the symbols $c_1, c_2, \ldots, c_k$ and the edges in cell E_r are said to have color c_r. This leads to the notion of *colored graph*, denoted by $\mathcal{G}(\pi) = (V, E, \pi)$. Furthermore, associated with $\mathcal{G}(\pi) = (V, E, \pi)$, we define the *colored qualitative class* $\mathcal{Q}_\pi(\mathcal{G})$ to describe the family of matrices associated with $\mathcal{G}(\pi)$:

$$\mathcal{Q}_\pi(\mathcal{G}) = \{A \in \mathcal{Q}(\mathcal{G}) \mid A_{ij} = A_{kl} \text{ if } (j,i),(l,k) \in E_r \text{ for some } r\}.$$

Example 1. *In this example, the edges in E_1 have color c_1 (blue), those in E_2 have color c_2 (green), and those in E_3 have color c_3 (red).*



$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & \lambda_4 & 0 & 0 \\ 0 & c_2 & c_3 & c_1 & \lambda_5 & c_3 \\ c_1 & 0 & c_3 & 0 & 0 & \lambda_6 \end{bmatrix} \in \mathcal{Q}_\pi(\mathcal{G})$$

The corresponding colored qualitative class consists of the matrices of the form as left above.

We call the system defined on $\mathcal{G}(\pi) = (V, E, \pi)$ with leader set $V_L \subset V$ *colored strongly structurally controllable* if the pair (A, B) is controllable for all $A \in \mathcal{Q}_\pi(\mathcal{G})$. For example, the system with graph depicted in Figure 1 will be shown to be colored strongly structurally controllable. We will use shorthand notation and say that $(\mathcal{G}(\pi); V_L)$ is controllable.

Fact 2. *Let $\mathcal{G}(\pi) = (V, E, \pi)$ be a colored simple directed graph, and let $V_L \subset V$ be a leader set. Then $(\mathcal{G}(\pi); V_L)$ is controllable if and only if V_L is a balancing set [3] for all weighted graphs $\mathcal{G}(W) = (V, E, W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$.*

Note that it is infeasible to test whether $(\mathcal{G}(\pi); V_L)$ is controllable by means of checking controllability of all systems $(\mathcal{G}(W); V_L)$ defined on weighted graphs related to the colored graph. Therefore, the aim of this paper is to establish graph-theoretic conditions under which $(\mathcal{G}(\pi); V_L)$ is controllable.

Colored Bipartite Graphs and Equivalence Classes

Consider an undirected bipartite graph $G = (X, Y, E_{XY})$, where the vertex sets X and Y are given by $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$. We will now introduce the notion of colored undirected bipartite graph. For this we introduce a partition of the edge set E_{XY} as $\pi_{XY} = \{E^1_{XY}, E^2_{XY}, \ldots, E^\ell_{XY},\}$ with associated colors $c_1, c_2, \ldots, c_\ell$.

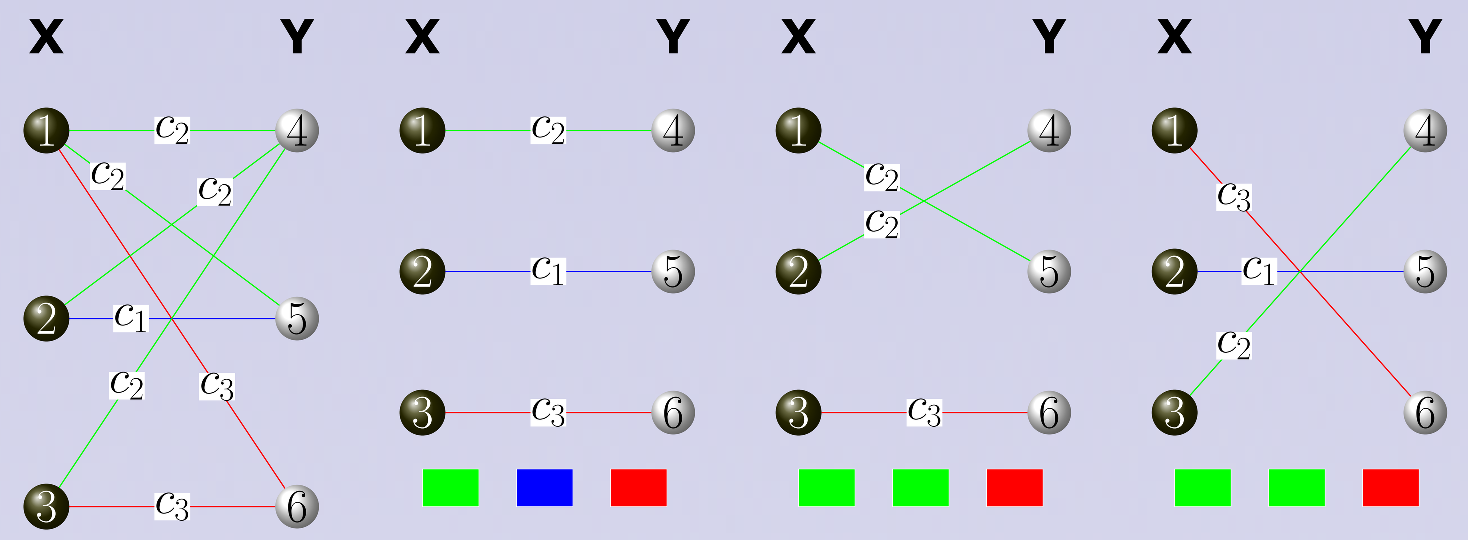


Figure 2: *Example of a colored undirected bipartite graph and its perfect matchings.*

The pattern class of the colored bipartite graph $G(\pi) = (X, Y, E_{XY}, \pi_{XY})$ is then defined as the following set of complex $t \times s$ matrices.

$$\mathcal{P}_\pi(G) = \{M \in \mathcal{P}(G) \mid M_{ji} = M_{hg} \text{ if } \{x_i, y_j\}, \{x_g, y_h\} \in E^r_{XY} \text{ for some } r\}.$$

Theorem 3. *Let $G(\pi) = (X, Y, E_{XY}, \pi_{XY})$ be a colored undirected bipartite graph and $|X| = |Y|$. Then, all $M \in \mathcal{P}_\pi(G)$ are nonsingular if and only if there exists at least one perfect matching, and exactly one equivalence class of perfect matchings has nonzero signature.* □

Colored Color Change Rule and Zero Forcing Sets

- Let $X \subset C$. Then $Y \subset V \setminus C$ is called a *color-perfect white neighbor* of X if $|X| = |Y|$, $Y = N_{V\setminus C}(X)$, and in the associated colored undirected bipartite graph $G = (X, Y, E_{XY}, \pi_{XY})$ there exists a perfect matching and exactly one equivalence class of perfect matchings has nonzero signature.
- If $Y \subset V \setminus C$ is a color-perfect white neighbor of X then X forces Y to be black, and we write $X \xrightarrow{C} Y$. Such step of coloring a white cell black will be called a *colored force*.
- The *colored derived set of* C, denoted by $D_c(C)$, is the set of black vertices obtained by applying the colored color change rule until no more changes are possible.
- A *colored zero forcing set* for $\mathcal{G}(\pi)$ is a subset Z of V such that $D_c(Z) = V$. For any colored zero forcing set Z of $\mathcal{G}(\pi)$, we can list the colored forces in the order that make all vertices colored black in the end.

Theorem 4. *Consider the colored simple directed graph $\mathcal{G}(\pi) = (V, E, \pi)$, with leader set $V_L \subset V$. Assume V_L is a colored zero forcing set. Then $(\mathcal{G}(\pi); V_L)$ is controllable.* □

Conclusions

In this paper we have studied strong structural controllability of leader/follower networks. In contrast to existing work in which the values of the nonzero off-diagonal entries of the matrices in the qualitative class are completely independent, in this work we have studied the situation that some of these entries are constrained to take identical values. This has been formalized using the concept of colored graph. We have introduced a new coloring rule and a new concept of zero forcing set. These have been used to formulate a sufficient condition for strong structural controllability of the colored graph. We have given an example showing that our sufficient condition is not necessary for controllability. In this paper we have formulated our main results without giving the proofs. These will be given in a future extended version of the paper.

References

[1] A. Chapman and M. Mesbahi. On strong structural controllability of networked systems: A constrained matching approach. In *2013 American Control Conference*, pages 6126--6131, June 2013.

[2] N. Monshizadeh, S. Zhang, and M. K. Camlibel. Zero forcing sets and controllability of dynamical systems defined on graphs. *IEEE Transactions on Automatic Control*, 59(9):2562--2567, Sept 2014.

[3] S. S. Mousavi, M. Haeri, and M. Mesbahi. On the structural and strong structural controllability of undirected networks. *IEEE Transactions on Automatic Control*, PP(99):1--1, 2017.